

THE (u, v) -CALKIN-WILF FOREST

SANDIE HAN, ARIANE M. MASUDA, SATYANAND SINGH, AND JOHANN THIEL

ABSTRACT. In this paper we consider a refinement, due to Nathanson, of the Calkin-Wilf tree. In particular, we study the properties of such trees associated with the matrices $L_u = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}$ and $R_v = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}$, where u and v are nonnegative integers. We extend several known results of the original Calkin-Wilf tree, including the symmetry, numerator-denominator, and successor formulas, to this new setting. Additionally, we study the ancestry of a rational number appearing in a generalized Calkin-Wilf tree.

1. INTRODUCTION

The Calkin-Wilf tree [4] is an infinite binary tree generated by two rules. The number 1, represented as $1/1$, is the root of the tree and each vertex a/b has two children: the left one is $a/(a+b)$ and the right one is $(a+b)/b$ (see Figure 1).

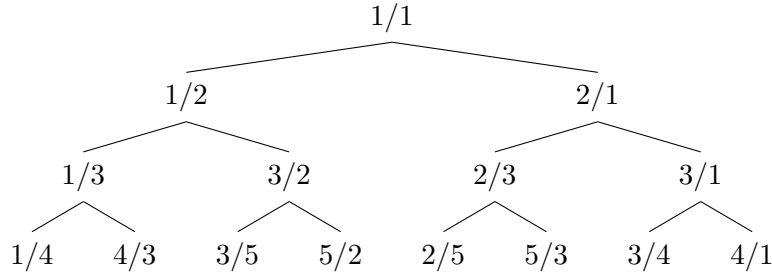


FIGURE 1. The first four rows of the Calkin-Wilf tree.

By following the breadth-first order, this tree provides an enumeration of positive rational numbers:

$$1, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \dots$$

In fact, Calkin and Wilf [4] showed that every reduced positive rational number appears in this list exactly once.

In addition to enumerating the positive rationals, the Calkin-Wilf tree has many interesting properties and generalizations that have been explored by various researchers (for example, [3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15]). In

particular, as in [12], we highlight the following four properties. We denote by $c(n, i)$ the vertex in the i^{th} position (from left to right) of the n^{th} row¹.

Property 1 (Successor formula, Newman [14]). *For every nonnegative integer n and $i = 1, \dots, 2^n - 1$, we have*

$$(1) \quad c(n, i+1) = \frac{1}{2[c(n, i)] + 1 - c(n, i)}$$

where $[x]$ denotes the integer part of x .

Property 2 (Denominator-numerator formula, Calkin and Wilf [4]). *For every nonnegative integer n and $i = 1, \dots, 2^n - 1$, the denominator of $c(n, i)$ is equal to the numerator of $c(n, i+1)$.*

Property 3 (Symmetry formula, [12]). *For every nonnegative integer n and $i = 1, \dots, 2^n$, we have $c(n, i) \cdot c(n, 2^n - i + 1) = 1$.*

Property 4 (Depth formula, [7]). *Let a/b be a positive reduced rational number. Let n and i be the unique pair such that $c(n, i) = a/b$. If*

$$\frac{a}{b} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{k-1} + \frac{1}{a_k}}}}} = [a_0, a_1, \dots, a_{k-1}, a_k]$$

is the finite continued fraction representation² of a/b , then

$$n = a_0 + a_1 + \dots + a_{k-1} + a_k - 1.$$

In other words, the sum of the coefficients of the continued fraction representation encodes the row number where a/b appears, i.e. the depth, in the Calkin-Wilf tree.

Let z be a variable. In [12], Nathanson considers the infinite binary tree $\mathcal{T}(z)$, whose root is z , where each vertex w has two children: the left child is $w/(w+1)$, and the right child is $w+1$ (see Figure 2).

The original Calkin-Wilf tree is clearly the special case of $z = 1$. For general z , Properties 1-4 of the Calkin-Wilf tree extend³ to $\mathcal{T}(z)$.

We can associate each vertex in $\mathcal{T}(z)$ with a column vector as in Figure 3.

Letting $L_1 := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $R_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we see that subsequent vertices

in $\mathcal{T}(z)$ can be obtained by matrix multiplication. A vertex $\begin{bmatrix} a \\ b \end{bmatrix}$ has left

¹Our convention is that the row containing the root is the zero row. So, for example, $c(2, 3) = 2/3$.

²For a rational number not equal to 1, we always take the shorter continued fraction representation where $a_k \neq 1$.

³Of independent interest, the generalization of Property 4 requires an appropriate definition of a continued fraction representation for linear fractional transformations.

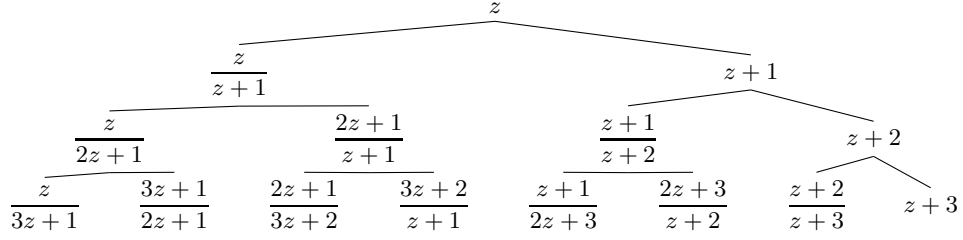
FIGURE 2. The first four rows of $\mathcal{T}(z)$.

FIGURE 3. Association between rational numbers and vectors.

child

$$(2) \quad L_1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a+b \end{bmatrix}$$

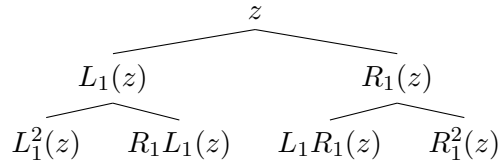
and right child

$$(3) \quad R_1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}.$$

In particular, every vertex in $\mathcal{T}(z)$ is obtained by multiplying a matrix generated freely by the set $\{L_1, R_1\}$ with the vector associated with z . In this way, we can label the vertices of $\mathcal{T}(z)$ with matrices in

$$SL_2(\mathbb{N}_0) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{N}_0 \text{ and } ad - bc = 1 \right\}$$

acting on z (see Figure 4). For ease of notation, we denote the left child and the right child of w by $L_1(w)$ and $R_1(w)$, respectively.

FIGURE 4. The first three rows of $\mathcal{T}(z)$ in terms of L_1 and R_1 .

With this perspective in mind, it is natural to consider an analogous infinite binary tree generated by other pairs of matrices in $SL_2(\mathbb{N}_0)$. Let u and v be integers such that $u, v \geq 2$,

$$L_u := \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} \quad \text{and} \quad R_v := \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}.$$

Nathanson [11, 12] proposed to investigate the infinite binary tree associated to $\{L_u, R_v\}$ obtained by replacing L_1 in (2) and R_1 in (3) by L_u and R_v , respectively (see Figure 5 for the generation rule).

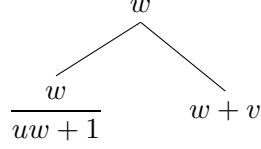


FIGURE 5. The children of w in $\mathcal{T}^{(u,v)}(z)$.

We refer to this generalization as a (u, v) -Calkin-Wilf tree and denote it by $\mathcal{T}^{(u,v)}(z)$, where z is the root (see Figure 6). Note that by setting $u = v = 1$ and $z = 1$, we obtain the original Calkin-Wilf tree, $\mathcal{T}(1)$. From now on, we assume that u and v are integers such that $u, v \geq 1$, and so $\mathcal{T}^{(1,1)}(1)$ is $\mathcal{T}(1)$.

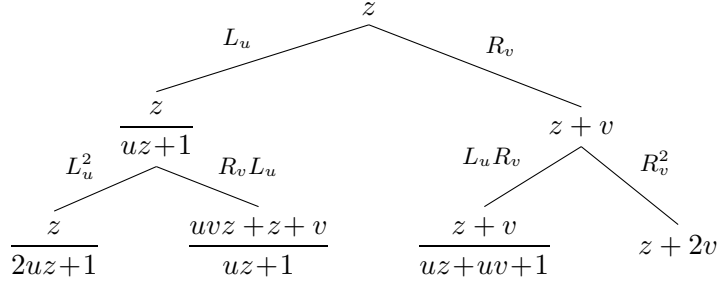
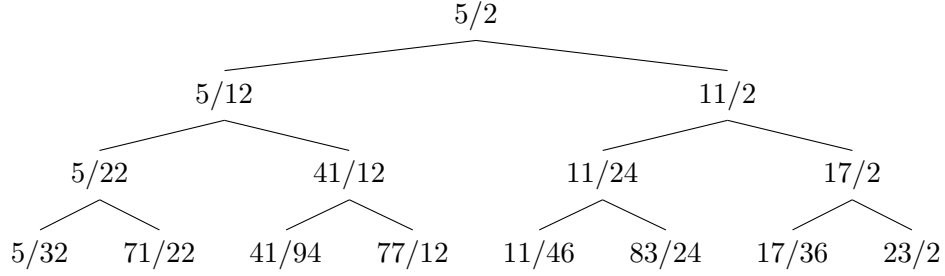


FIGURE 6. The first three rows of $\mathcal{T}^{(u,v)}(z)$.

As an example, consider the tree $\mathcal{T}^{(2,3)}(5/2)$ (see Figure 7). One can immediately notice that the denominator-numerator and the symmetry formulas (Properties 2 and 3) do not hold in $\mathcal{T}^{(2,3)}(5/2)$. Furthermore, many rational numbers appearing in $\mathcal{T}(1)$ seem to be missing in $\mathcal{T}^{(2,3)}(5/2)$. In fact, it is not too difficult to show that 1 does not appear in any tree $\mathcal{T}^{(u,v)}(z)$ unless $z = 1$. In the next section we will address this issue, and define the (u, v) -Calkin-Wilf forest which will enumerate positive rational numbers.

We have already shown by example that Properties 1-4 do not, in general, hold for a (u, v) -Calkin-Wilf tree. However, (u, v) -Calkin-Wilf trees share enough of a similar structure with $\mathcal{T}(1)$ that we are able to provide some

FIGURE 7. The first four rows of $\mathcal{T}^{(2,3)}(5/2)$.

appropriate, universal generalizations (see Theorem 1 and Corollary 3, for example). In other cases, we will show that some of Properties 1-4 completely characterize the Calkin-Wilf tree (see Proposition 4 and Corollary 1, for example).

2. GLOBAL PROPERTIES

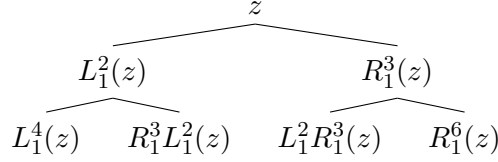
For a fixed u and v , consider the set of all positive reduced rational numbers that are not the children of any rational number appearing in *any* (u, v) -Calkin-Wilf tree. We refer to such numbers as (u, v) -orphans (when the context is clear, we may refer to such numbers simply as orphans). A straightforward proof shows that the set of (u, v) -orphans is

$$\left\{ \frac{a}{b} : \frac{1}{u} \leq \frac{a}{b} \leq v \right\}$$

(see [12]). It follows that the set of (u, v) -orphans is finite if and only if $u = v = 1$. Furthermore, it can be seen that every left child in a (u, v) -Calkin-Wilf tree is strictly bounded above by $1/u$ and every right child is strictly bounded below by v . In the case of the original Calkin-Wilf tree 1 is the only orphan. In $\mathcal{T}^{(2,3)}(5/2)$, the vertex $5/2$ satisfies the condition $1/2 \leq 5/2 \leq 3$, and so it is one of the many $(2, 3)$ -orphans.

Lemma 1. *Let z and z' be distinct (u, v) -orphans. Then the vertices of $\mathcal{T}^{(u,v)}(z)$ and $\mathcal{T}^{(u,v)}(z')$ form disjoint sets.*

Proof. Suppose that w is a rational number that appears as a vertex in both $\mathcal{T}^{(u,v)}(z)$ and $\mathcal{T}^{(u,v)}(z')$. Without loss of generality, we can assume that w is such that no other ancestor of it (in either tree) holds this property. It follows w is not a root and must be the child of vertices in both trees. Furthermore, w cannot be a left child (right child, resp.) in both $\mathcal{T}^{(u,v)}(z)$ and $\mathcal{T}^{(u,v)}(z')$. So w is a left child in, say, $\mathcal{T}^{(u,v)}(z)$ and a right child in $\mathcal{T}^{(u,v)}(z')$. This implies that $w < 1/u \leq 1$ and $w > v \geq 1$, a contradiction. \square

FIGURE 8. The first three rows of $\mathcal{T}^{(2,3)}(z)$ in terms of L_1 and R_1 .

Since every positive reduced rational number is either a (u, v) -orphan or the descendant of a (u, v) -orphan, Lemma 1 shows that the set of (u, v) -orphans enumerates a forest of trees that partitions the set of positive rational numbers, the (u, v) -Calkin-Wilf forest.

Lemma 2. *Let u and v be positive integers. Then $L_u = L_1^u$ and $R_v = R_1^v$.*

Proof. We show that $L_u = L_1^u$ by induction on u . This is clearly true when $u = 1$. Suppose it is true for $u \geq 1$. Then

$$L_1^{u+1} = L_1^u \cdot L_1 = L_u \cdot L_1 = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u+1 & 1 \end{bmatrix} = L_{u+1}.$$

A similar argument shows that $R_v = R_1^v$. \square

When comparing $\mathcal{T}^{(u,v)}(z)$ to $\mathcal{T}(1)$, one can see from Lemma 2 that the vertices of $\mathcal{T}^{(u,v)}(z)$ can be obtained by starting with the vertex z in $\mathcal{T}(1)$ and skipping over $u - 1$ generations of left children or $v - 1$ generations of right children to arrive at the children of z in $\mathcal{T}^{(u,v)}(z)$. For example, compare Figure 4 and Figure 8 in the case where $u = 2$ and $v = 3$. In other words, the vertex set of $\mathcal{T}^{(u,v)}(z)$ is a submonoid of \mathbb{Q} or, equivalently, the vertex set of $\mathcal{T}(1)$. More generally, we have the following two results as other immediate consequences of Lemma 2.

Proposition 1. *The vertex set of $\mathcal{T}^{(u,v)}(z)$ is a submonoid of the vertex set of $\mathcal{T}^{(u',v')}(z')$ if and only if $z \in \mathcal{T}^{(u',v')}(z')$, $u' \mid u$, and $v' \mid v$.*

Proposition 2. *Let U and V be finite sets of nonnegative integers. Set $u := \text{lcm}\{u' : u' \in U\}$ and $v := \text{lcm}\{v' : v' \in V\}$. Then*

$$\mathcal{T}^{(u,v)}(z) = \bigcap_{u' \in U, v' \in V} \mathcal{T}^{(u',v')}(z).$$

Lemma 1 and Lemma 2 show that the (u, v) -orphans partition the set of positive rational numbers into a collection of trees with a similar structure derived from $\mathcal{T}(1)$. This idea serves as the main motivation for this paper.

3. THE SUCCESSOR AND THE NUMERATOR-DENOMINATOR FORMULAS

We begin this section by establishing some immediate properties of (u, v) -Calkin-Wilf trees related to Properties 1-2. We denote by $c_z^{(u,v)}(n, i)$ the i^{th} element, from left to right, in the n^{th} row of the (u, v) -Calkin-Wilf tree

whose root is z . For any integer $n \geq 0$, the first and the last elements of the n^{th} row with root z are readily seen to be

$$c_z^{(u,v)}(n, 1) = \frac{z}{nuz + 1} \quad \text{and} \quad c_z^{(u,v)}(n, 2^n) = z + nv,$$

respectively. Furthermore, since z is assumed to be in reduced form, then all vertices of $\mathcal{T}^{(u,v)}(z)$ are also in reduced form.

Proposition 3 (Generalized successor formula). *Consider the (u, v) -Calkin-Wilf tree with root z . For every nonnegative integer n and $i = 1, \dots, 2^n - 1$, let $\alpha_i = c_z^{(u,v)}(n, i)$. Then we have*

$$(4) \quad \alpha_{i+1} = \frac{v\{\alpha_i\} + v^2(1 - u\{\alpha_i\})}{u[\alpha_i](\{\alpha_i\} + v(1 - u\{\alpha_i\})) + v(1 - u\{\alpha_i\})}$$

where $[x]$ and $\{x\}$ denote the integer and fractional parts of the real number x , respectively.

Proof. Our proof is a generalization of an argument by Newman [1] in the case where $u = v = 1$.

If α_i and α_{i+1} are adjacent siblings in a (u, v) -Calkin-Wilf tree, then they share a common ancestor w (see Figure 9) such that, for some $k \geq 0$, α_i is the k^{th} right child of the left child of w and α_{i+1} is the k^{th} left child of the right child of w . (This is not a feature that is unique to (u, v) -Calkin-Wilf trees; it is common to all full binary trees.) It follows that $\alpha_i = \frac{w}{uw+1} + kv$ and $\alpha_{i+1} = \frac{w+v}{ku(w+v)+1}$. Note that since $\frac{w}{uw+1} < 1$, then $\{\alpha_i\} = \frac{w}{uw+1}$ and $[\alpha_i] = kv$.

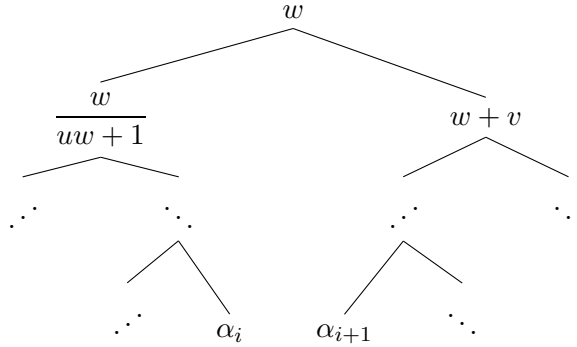


FIGURE 9. Successors in a (u, v) -Calkin-Wilf tree with common ancestor w .

In order to complete the proof, we must eliminate the dependence of α_{i+1} on k and w . This can be accomplished by taking the formula for $\{\alpha_i\}$ and solving for w . This gives that

$$(5) \quad w = \frac{\{\alpha_i\}}{1 - u\{\alpha_i\}}.$$

It follows that

$$(6) \quad \alpha_{i+1} = \frac{w+v}{ku(w+v)+1} = \frac{w+v}{kv(\frac{wu}{v}+u)+1} = \frac{w+v}{[\alpha_i](\frac{wu}{v}+u)+1}.$$

Inserting (5) into the right-hand side of (6) and simplifying gives the desired result. \square

While (4) does collapse down to (1) when $u = v = 1$, something is lost in this generalization. Iterating (1) not only gives successive elements in a fixed row of the Calkin-Wilf tree; when it is applied to the rightmost element of a row, it returns the leftmost element of the next row. The same is not true of (4).

It follows from Proposition 3 that if we consider successive terms in each row of a (u, v) -Calkin-Wilf tree, the denominator-numerator formula (Property 2) holds only in the original Calkin-Wilf tree.

Proposition 4. *The denominator-numerator formula holds if and only if $u = v = 1$.*

Proof. Using the same notation in the proof of Proposition 3, for a common ancestor w ,

$$\alpha_i = \frac{w' + kv(uw' + w'')}{uw' + w''} \quad \text{and} \quad \alpha_{i+1} = \frac{w' + vw''}{ku(w' + vw'') + w''},$$

where $w = w'/w''$ is in lowest terms. It is easy to see that the above representations of α_i and α_{i+1} are also in lowest terms. So we can let $d_i = uw' + w''$ be the denominator of α_i and $n_{i+1} = w' + vw''$ be the numerator of α_{i+1} . It quickly follows that

$$(7) \quad vd_i + (1 - uv)w' = n_{i+1}.$$

(\Leftarrow) If $u = v = 1$, then $d_i = n_{i+1}$ follows from (7).

(\Rightarrow) If $d_i = n_{i+1}$, then it follows from (7) that

$$(uv - 1)w' = (v - 1)n_{i+1} = (v - 1)(w' + vw'').$$

Collecting like terms on either side of the equality shows that $(u - 1)w' = (v - 1)w''$. If $u = 1$ and $v \neq 1$, then $w'' = 0$, a contradiction. A similar argument works for the case where $u \neq 1$ and $v = 1$. If $u, v \neq 1$, then $w = w'/w'' = (v - 1)/(u - 1)$. This would imply that w is fixed for all pairs of successors, another contradiction. Therefore, $u = v = 1$. \square

We see from (7) that the relationship between successive denominators and numerators in a row of a (u, v) -Calkin-Wilf tree is significantly more complicated than in the statement of Property 2. In order to generalize the denominator-numerator formula, one would need to know more about the common ancestors of successive terms. At this time, no clear generalization of Property 2 is evident.

4. SYMMETRY PROPERTIES

In this section, we study symmetry properties of (u, v) -Calkin-Wilf trees closely related to Property 3. As in the previous section, we are able to find some appropriate generalizations, in some sense, while showing that Property 3 completely characterizes $\mathcal{T}(1)$. We begin with a lemma which will be used in the theorems that follow.

Lemma 3. *For every vertex in the (u, v) -Calkin-Wilf tree with root z there are nonnegative integers a, b, c , and d with $ad - bc = 1$ such that the vertex is represented as $\frac{az + b}{cz + d}$.*

Proof. The statement follows from induction on the row number of the (u, v) -Calkin-Wilf tree $\mathcal{T}^{(u,v)}(z)$ (see Figure 6). \square

Note that the integers a, b, c , and d in Lemma 3 depend on u, v , and the position of the vertex in the tree. See (14) in Section 5 for an example on how to compute a, b, c , and d for $2147/620$ in $\mathcal{T}^{(2,3)}(5/2)$. Furthermore, Lemma 3 shows that every vertex in a (u, v) -Calkin-Wilf tree can be written as some linear fractional transformation of the root (see Figures 6 and 10).

Theorem 1 (General symmetry formula). *For every nonnegative integer n and $i = 1, 2, \dots, 2^n$, if $c_z^{(u,v)}(n, i) = \frac{az + b}{cz + d}$ where a, b, c, d are nonnegative integers, then*

$$(8) \quad c_z^{(u,v)}(n, 2^n + 1 - i) = \frac{dz + \frac{cv}{u}}{\frac{bu}{v}z + a}.$$

Proof. The proof is by induction on the row number n . Since $c_z^{(u,v)}(1, 1) = \frac{z}{uz + 1}$, we have that $c_z^{(u,v)}(1, 1) = \frac{az + b}{cz + d}$ with $a = 1, b = 0, c = u$, and $d = 1$, and so

$$\frac{dz + \frac{cv}{u}}{\frac{bu}{v}z + a} = z + v = c_z^{(u,v)}(1, 2).$$

On the other hand, starting from $c_z^{(u,v)}(1, 2) = z + v$, we get that $c_z^{(u,v)}(1, 2) = \frac{az + b}{cz + d}$ with $a = 1, b = v, c = 0$, and $d = 1$. Hence

$$\frac{dz + \frac{cv}{u}}{\frac{bu}{v}z + a} = \frac{z}{uz + 1} = c_z^{(u,v)}(1, 1).$$

This shows that the statement is true when $n = 1$. Suppose that the theorem is true for some row $n \geq 1$. An element in the row $n + 1$ is either of the form $c_z^{(u,v)}(n + 1, 2i - 1)$ or $c_z^{(u,v)}(n + 1, 2i)$ for some integer $i, 1 \leq i \leq 2^n$. If

$c_z^{(u,v)}(n+1, 2i-1) = \frac{az+b}{cz+d}$ (we know that such a representation exists by Lemma 3) then it is the left child of

$$c_z^{(u,v)}(n, i) = \frac{az+b}{(c-ua)z+(d-ub)}.$$

Thus, by using the symmetry on row n , we obtain

$$\begin{aligned} c_z^{(u,v)}(n+1, 2^{n+1}+2-2i) &= R_v \left(c_z^{(u,v)}(n, 2^n+1-i) \right) \\ &= R_v \left(\frac{(d-ub)z + \frac{(c-ua)v}{u}}{\frac{bu}{v}z + a} \right) = \frac{dz + \frac{cv}{u}}{\frac{bu}{v}z + a}. \end{aligned}$$

Similarly, if $c_z^{(u,v)}(n+1, 2i) = \frac{az+b}{cz+d}$ then it is the right child of

$$c_z^{(u,v)}(n, i) = \frac{(a-cv)z + (b-vd)}{cz+d}.$$

Hence

$$\begin{aligned} c_z^{(u,v)}(n+1, 2^{n+1}+1-2i) &= L_u \left(c_z^{(u,v)}(n, 2^n+1-i) \right) \\ &= L_u \left(\frac{dz + \frac{cv}{u}}{\frac{(b-vd)u}{v}z + (a-cv)} \right) = \frac{dz + \frac{cv}{u}}{\frac{bu}{v}z + a}. \end{aligned}$$

□

As a consequence, we obtain necessary and sufficient conditions for the symmetry formula (Property 3) to hold in a (u, v) -Calkin-Wilf tree.

Corollary 1 (Symmetry formula). *The symmetry formula,*

$$(9) \quad c_z^{(u,v)}(n, i) \cdot c_z^{(u,v)}(n, 2^n+1-i) = 1,$$

holds if and only if $u = v$ and $z = 1$.

Proof. Suppose, using Lemma 3, that $c_z^{(u,v)}(n, i) = \frac{az+b}{cz+d}$ where a, b, c, d are nonnegative integers and $ad - bc = 1$. By Theorem 1, we obtain that (9) is equivalent to

$$\left(\frac{az+b}{cz+d} \right) \cdot \left(\frac{dz + \frac{cv}{u}}{\frac{bu}{v}z + a} \right) = 1,$$

or

$$(10) \quad \left(ad - \frac{bcu}{v} \right) z^2 + \left[bd \left(1 - \frac{u}{v} \right) - ac \left(1 - \frac{v}{u} \right) \right] z + \left(\frac{bcv}{u} - ad \right) = 0$$

It follows that $ad - \frac{bcu}{v} = 0$ and $\frac{bcv}{u} - ad = 0$, from which we get

$$\frac{v}{u} = \frac{bc}{ad} \quad \text{and} \quad \frac{v}{u} = \frac{ad}{bc},$$

thus $v^2 = u^2$. Since $u, v > 0$, this implies that $u = v$. By substituting $u = v$ into (10), we obtain that $(ad - bc)(z^2 - 1) = 0$. Since $ad - bc = 1$ and $z > 0$, we conclude that $z = 1$. \square

We remark that Corollary 1 can be also proved using induction on the row number. The result explains why the symmetry formula does not hold in $\mathcal{T}^{(2,3)}(5/2)$ (see Figure 7), as we had observed earlier.

Corollary 2 (Skew symmetry). *Using the same hypothesis as Theorem 1, it follows that*

$$c_z^{(u,v)}(n, i) \cdot c_{\frac{v}{u}z^{-1}}^{(u,v)}(n, 2^n + 1 - i) = \frac{v}{u}.$$

Proof. Suppose, using Lemma 3, that $c_z^{(u,v)}(n, i) = \frac{az + b}{cz + d}$ where a, b, c, d are nonnegative integers. Replacing z by $\frac{v}{uz}$ in (8) yields that

$$c_{\frac{v}{u}z^{-1}}^{(u,v)}(n, 2^n + 1 - i) = \frac{d\frac{v}{uz} + \frac{cv}{u}}{\frac{bu}{v}\frac{v}{uz} + a} = \frac{\frac{v}{u}(d + cz)}{b + az} = \frac{v}{u} \cdot \frac{cz + d}{az + b},$$

which is equivalent to the desired result. \square

Corollary 1 shows that the symmetry formula does not hold for (u, v) -Calkin-Wilf trees in general. However, Corollary 2 (above) and Theorem 2 (below) show that other symmetry formulas do hold when comparing either pairs of (u, v) -Calkin-Wilf trees or (u, v) - and (v, u) -Calkin-Wilf trees, respectively. For examples, see Table 1.

Row 2 of $\mathcal{T}^{(2,3)}(5/2)$	5/22	41/12	11/24	17/2
Row 2 of $\mathcal{T}^{(2,3)}(3/5)$	3/17	36/11	18/41	33/5
Row 2 of $\mathcal{T}^{(3,2)}(2/5)$	2/17	24/11	12/41	22/5

TABLE 1. Examples of Corollary 2 and Theorem 2

Theorem 2 (Nathanson's symmetry, [13]). *Let z be a variable, and let u and v be positive integers. For all nonnegative integers n and $i = 1, 2, \dots, 2^n$,*

$$c_z^{(u,v)}(n, i) \cdot c_{z^{-1}}^{(v,u)}(n, 2^n + 1 - i) = 1.$$

If $u = v \geq 1$, then Theorem 2 gives one of the directions of Corollary 1. If $u = v = 1$, then this is the familiar symmetry of the Calkin-Wilf tree.

Nathanson's symmetry was proved in [13] using induction on the row number. We conclude this section with two alternative proofs of Theorem 2. The first one is a consequence of Theorem 1, and only holds when $u = v$.

First Proof of Theorem 2 when $u = v$. By Lemma 3, let $c_z^{(u,u)}(n, i) = \frac{az + b}{cz + d}$ for some nonnegative integers a, b, c , and d . By Theorem 1, we have

$$c_z^{(u,u)}(n, 2^n + 1 - i) = \frac{dz + c}{bz + a} \implies c_{z^{-1}}^{(u,u)}(n, 2^n + 1 - i) = \frac{dz^{-1} + c}{bz^{-1} + a} = \frac{cz + d}{az + b},$$

which is the reciprocal of $c_z^{(u,u)}(n, i)$. \square

The identity presented in Theorem 1 only holds in a (u, v) -Calkin-Wilf tree where u and v are fixed. Therefore we cannot use it to derive Nathanson's symmetry in the case $u \neq v$. In order to show the desired relationship between (u, v) - and (v, u) -Calkin-Wilf trees, we will use a lemma that shows the following:

Lemma 4. *Let $\sigma : \mathbb{Q}^* \rightarrow \mathbb{Q}^*$ be defined by $\sigma(x) = x^{-1}$. Then*

- (a) $\sigma \circ L_u \circ \sigma = R_u$
- (b) $\sigma \circ R_u \circ \sigma = L_u$

Proof. Part (a) of the lemma follows from the following straightforward computation:

$$(\sigma \circ L_u \circ \sigma)(x) = \sigma\left(\frac{x^{-1}}{ux^{-1} + 1}\right) = \sigma\left(\frac{1}{u + x}\right) = x + u = R_u(x).$$

Part (b) follows from (a) since $\sigma^2 = id$ \square

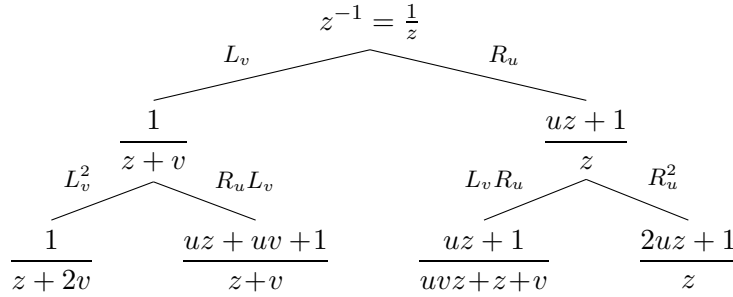


FIGURE 10. The first three rows of $\mathcal{T}^{(v,u)}(z^{-1})$.

Comparing Figures 6 and 10, we can see that if we view $c_z^{(u,v)}(n, i)$ as the result of a (unique) word $w(L_u, R_v)$ on two letters acting on z , then $c_z^{(v,u)}(n, 2^n + 1 - i) = w(R_u, L_v)(z)$. Specifically, the vertex $c_z^{(u,v)}(n, i)$ is $w(L_u, R_v)(z)$ where w is the i^{th} word of length n on the letters R_u and L_v in the reverse lexicographic order. We will use this approach to prove Nathanson's symmetry in its general form.

Second Proof of Theorem 2. Let $c_z^{(u,v)}(n, i) = w(L_u, R_v)(z)$ where w is the i^{th} word of length n on the letters R_u and L_v in the reverse lexicographic order. For $\sigma(z) = z^{-1}$, it follows from Lemma 4 that

$$\sigma \circ w(L_u, R_v) \circ \sigma = w(\sigma \circ L_u \circ \sigma, \sigma \circ R_v \circ \sigma) = w(R_u, L_v).$$

Therefore $\sigma \circ w(L_u, R_v) = w(R_u, L_v) \circ \sigma$, which means that $\left(c_z^{(u,v)}(n, i)\right)^{-1} = c_{z^{-1}}^{(v,u)}(n, 2^n + 1 - i)$. \square

5. THE DESCENDANT CONDITIONS AND THE DEPTH FORMULA

In a full binary tree, each vertex can be assigned a binary representation by enumerating the vertices in a breadth-first order. For example, the root of the tree is assigned the number 1; its left child is 2 and right child is 3, or 10_2 and 11_2 in their respective binary representations. In the next row, the vertices are 4, 5, 6, 7, or 100_2 , 101_2 , 110_2 , 111_2 , in binary representation form (See Figure 11).

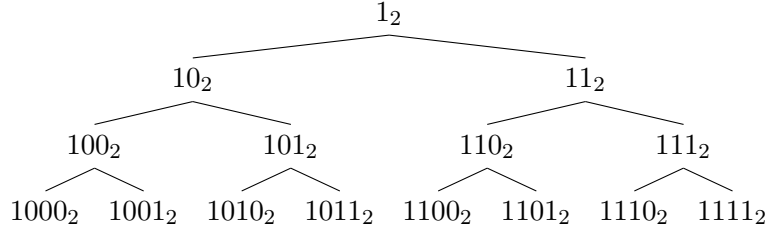


FIGURE 11. Binary representation tree.

The parent-child relation is clearly demonstrated by the binary representation. Each left child is represented by the binary representation of its parent followed by a 0, while each right child is represented by the binary representation of its parent followed by a 1. Moreover, for each vertex, its binary representation encodes the binary representations of all of its ancestors back to the root.

We construct a 1-1 correspondence between the binary representations of the vertices in a full-binary tree and the words associated with each vertex in the Calkin-Wilf tree. Begin with the binary representation of a vertex. Truncate the leftmost 1 digit (all such representations begin with a 1), reverse the order of the string and map $0 \mapsto L_1$ and $1 \mapsto R_1$. For example, the vertex in position 1100_2 corresponds to the word $L_1^2 R_1$, which corresponds to the number $2/5 = L_1^2 R_1(1)$ in the Calkin-Wilf tree.

In the (u, v) -Calkin-Wilf tree, if we use the same binary representation as those in the original Calkin-Wilf tree, we can easily see that the left child is represented by the binary representation of its parent followed by u consecutive 0s and the right child is represented by the binary representation of its parent followed by v consecutive 1s. Let B be the binary representation

of the position of w in the original Calkin-Wilf tree. Figure 12 shows the first three rows of $\mathcal{T}^{(2,3)}(w)$ in binary form.

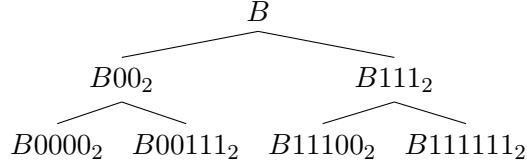


FIGURE 12. Binary representation tree for $\mathcal{T}^{(2,3)}(w)$.

The (u, v) -ancestor-descendant relation is clearly demonstrated by the sequence of u consecutive 0s or v consecutive 1s. We give a few examples related to $(2, 3)$ -Calkin-Wilf trees:

- We have that $2/5 \mapsto 1100_2$, which is the left child of $11_2 \mapsto 2$. Incidentally, 2 is an orphan root in the $(2, 3)$ -Calkin-Wilf forest.
- The rational number corresponding to 110001110000_2 in the Calkin-Wilf tree is a descendant of the orphan root 110_2 . One can trace from the right, a sequence of four 0s, three 1s, two 0s, and then it offers neither two consecutive 0s nor three consecutive 1s.
- The rational number corresponding to the position 110001110001_2 in the Calkin-Wilf tree is an orphan in the $(2, 3)$ -Calkin-Wilf forest.

The following result formalizes the above criterion for an element to be an orphan or a child of a (u, v) -Calkin-Wilf tree.

Proposition 5. *Let w be a vertex of a (u, v) -Calkin-Wilf tree, and $B(w)$ be the binary representation of its corresponding position in the original Calkin-Wilf tree.*

- (a) Suppose that $B(w) = B_1 \underbrace{0 \dots 0}_i$, i.e., the binary representation $B(w)$ ends in exactly i 0s. If $i \geq u$, then w is the left child of the vertex whose position is $B_1 \underbrace{0 \dots 0}_{i-u}$. Otherwise, w is an orphan.
- (b) Suppose that $B(w) = B_0 \underbrace{1 \dots 1}_j$, i.e., the binary representation $B(w)$ ends in exactly j 1s. If $j \geq v$, then w is the right child of the vertex whose position is $B_0 \underbrace{1 \dots 1}_{j-v}$. Otherwise, w is an orphan.

Another viewpoint for understanding the relationship between descendants in a (u, v) -Calkin-Wilf tree is via continued fractions. We begin the study of the relationship between continued fractions and (u, v) -Calkin-Wilf trees with the following useful lemma (see [2] for the case $u = v = 1$).

Lemma 5 (Continued fraction relationship). *Let $\frac{a}{b}$ be a positive rational number with continued fraction representation $\frac{a}{b} = [q_0, q_1, \dots, q_r]$. It follows that*

- (a) *if $q_0 = 0$, then $\frac{a}{ua+b} = [0, u + q_1, \dots, q_r]$;*
- (b) *if $q_0 \neq 0$, then $\frac{a}{ua+b} = [0, u, q_0, q_1, \dots, q_r]$;*
- (c) *and $\frac{a+vb}{b} = [v + q_0, q_1, \dots, q_r]$.*

Proof. Let

$$\frac{a}{b} = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{\ddots + \frac{1}{q_{r-1} + \frac{1}{q_r}}}}}.$$

Note that $\frac{a}{ua+b} = \left(u + \frac{b}{a}\right)^{-1}$, so

$$(11) \quad \frac{a}{ua+b} = \frac{1}{u + \frac{1}{q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{\ddots + \frac{1}{q_{r-1} + \frac{1}{q_r}}}}}}}.$$

By considering the cases when $q_0 = 0$ and $q_0 \neq 0$, we get (a) and (b). The remaining case follows from the fact that $\frac{a+vb}{b} = \frac{a}{b} + v$. \square

Lemma 5 shows that the continued fraction representations of rationals appearing in a (u, v) -Calkin-Wilf tree follow a nice pattern. In fact, in the case where $u = v = 1$, we can recover several of the properties of the original Calkin-Wilf tree listed in Section 1.

The next theorem gives more insight into the properties of coefficients in the continued fraction representation of rational numbers appearing in a (u, v) -Calkin-Wilf tree.

Theorem 3 (Descendant conditions). *Suppose that w and w' are positive rational numbers with continued fraction representations $w = [q_0, q_1, \dots, q_r]$ and $w' = [p_0, p_1, \dots, p_s]$. Then w' is a descendant of w in the (u, v) -Calkin-Wilf tree with root w if and only if the following conditions all hold:*

- (a) $s \geq r$ and $2 \mid (s - r)$;
- (b) for $0 \leq j \leq s - r - 1$, $v \mid p_j$ when j is even and $u \mid p_j$ when j is odd;
- (c) for $2 \leq i \leq r$, $p_{s-r+i} = q_i$;
- (d) and
 - (i) if $q_0 \neq 0$, then $p_{s-r} \geq q_0$, $v \mid (p_{s-r} - q_0)$ and $p_{s-r+1} = q_1$;
 - (ii) otherwise, if $q_0 = 0$, then $v \mid p_{s-r}$, $p_{s-r+1} \geq q_1$, and $u \mid (p_{s-r+1} - q_1)$.

Proof. (\Rightarrow) We prove the first direction by induction. Note that (a) holds by Lemma 5, so our main concern will involve the remaining conditions.

Let A_n be the set of descendants of w of depth n . Then A_1 consists of both children of w . If w' is the left child of w and $q_0 = 0$, then, by Lemma 5, w' has a continued fraction representation $w' = [0, u + q_1, \dots, q_r]$. In this case, $s = r$, so (b) is vacuously true and (c) immediately holds. (Note that (c) is also vacuously true if $r = 1$.) Since $s - r = 0$, it follows that $p_{s-r} = p_0 = q_0$, which implies that $v \mid (p_{s-r} - q_0)$. Also, it is clear that $p_{s-r+1} \geq q_1$ and $u \mid (p_{s-r+1} - q_1)$ since $p_{s-r+1} = u + q_1$. This shows that part (iii) of condition (d) holds. The two remaining cases, where w' is a left child of w with $q_0 \neq 0$ and where w' is a right child of w , can be handled in a similar way using Lemma 5. This shows that the theorem holds for A_1 .

Now suppose that the desired result holds for A_k for some $k \geq 1$ and assume that $w' \in A_{k+1}$. Furthermore, assume that w' is the left child of some $w'' \in A_k$, where w'' has a continued fraction representation $w'' = [d_0, d_1, \dots, d_t]$. By Lemma 5, if $d_0 = 0$, then $s = t$ and $w' = [0, u + d_1, \dots, d_t]$. Since $p_k = d_k$ for $0 \leq k \leq t$ with $k \neq 1$, then, with the exception of one coefficient, the result holds. For the case $k = 1$, notice that if $t > r$, then $u \mid d_1$, so $u \mid (u + d_1)$. If $t = r$, then $u + d_1 - q_1 > d_1 - q_1 \geq 0$ and $u \mid (d_1 - q_1)$, so $u \mid (u + d_1 - q_1)$. This implies the desired result.

As was the case with A_1 , there are two remaining cases to handle. The proofs of the statement when $d_0 \neq 0$ and when w' is the right child of some $w'' \in A_k$ are both similar to the argument presented above. We omit the details.

(\Leftarrow) Using Lemma 5, a simple computation shows that when $q_0 \neq 0$,

$$(12) \quad w' = R_v^{p_0/v} L_u^{p_1/u} \dots R_v^{p_{s-r-2}/v} L_u^{p_{s-r-1}/u} R_v^{(p_{s-r}-q_0)/v}(w).$$

A similar formula gives the desired result when $q_0 = 0$. \square

Corollary 3 (Depth formula). *Using the same hypothesis as Theorem 3, if n is the depth of w' , then*

$$(13) \quad n = \frac{1}{v} \left(\sum_{\substack{0 \leq j \leq s-r-1 \\ j \text{ even}}} p_j + \sum_{\substack{0 \leq i \leq r \\ i \text{ even}}} (p_{s-r+i} - q_i) \right) + \frac{1}{u} \left(\sum_{\substack{0 \leq j \leq s-r-1 \\ j \text{ odd}}} p_j + \sum_{\substack{0 \leq i \leq r \\ i \text{ odd}}} (p_{s-r+i} - q_i) \right).$$

The proof of Corollary 3 follows from Theorem 3 by induction. Note that the majority of the terms in the sum (13) are actually zero. In the case where $u = v = 1$, Corollary 3 recovers the formula from Property 4.

From Lemma 5 and Theorem 3, we can construct a recursive algorithm that determines the orphan ancestor of w' in the (u, v) -Calkin-Wilf that contains it. The algorithm makes heavy use of the continued fraction representation of w' .

Algorithm 1 (u, v) -Calkin-Wilf tree orphan ancestor

```

1: procedure ANCESTOR( $[p_0, p_1, \dots, p_s], u, v$ )
2:   if  $s = 0$  then
3:     if  $p_0 \leq v$  then return  $[p_0]$ 
4:     else return ANCESTOR( $[p_0 - v], u, v$ )
5:     end if
6:   else if  $s = 1$  then
7:     if  $0 < p_0 < v$  then return  $[p_0, p_1]$ 
8:     else if  $p_0 > v$  then
9:       return ANCESTOR( $[p_0 - v, p_1], u, v$ )
10:    else if  $p_0 = 0$  and  $p_1 \leq u$  then return  $[0, p_1]$ 
11:    else return ANCESTOR( $[0, p_1 - u], u, v$ )
12:    end if
13:  else
14:    if  $p_0 < v$  then return  $[p_0, p_1, \dots, p_s]$ 
15:    else if  $p_0 \geq v$  then
16:      return ANCESTOR( $[p_0 - v, p_1, \dots, p_s], u, v$ )
17:    else if  $p_0 = 0$  and  $0 < p_1 < u$  then
18:      return ANCESTOR( $[0, p_1, \dots, p_s], u, v$ )
19:    else if  $p_0 = 0$  and  $p_1 > u$  then
20:      return ANCESTOR( $[0, p_1 - u, \dots, p_s], u, v$ )
21:    else return ANCESTOR( $[p_2, \dots, p_s], u, v$ )
22:    end if
23:  end if
24: end procedure

```

For example, let $u = 2$ and $v = 3$. The continued fraction representation of $2147/620$ is given by $[3, 2, 6, 4, 5, 2]$. Using the above algorithm, we can compute the list of ancestors of $2147/620$ as: $287/620 = [0, 2, 6, 4, 5, 2]$, $287/46 = [6, 4, 5, 2]$, $149/46 = [3, 4, 5, 2]$, $11/46 = [0, 4, 5, 2]$, $11/24 = [0, 2, 5, 2]$, $11/2 = [5, 2]$, and $5/2 = [2, 2]$. Since $1/2 \leq 5/2 \leq 3$, then $5/2$ is the orphan ancestor of $2147/620$.

By (12), we see that the coefficients of the continued fraction of $2147/620$ encode the path taken from the orphan $5/2$ to the descendant $2147/620$. This can be computed as follows. Consider the continued fraction representation $[3, 2, 6, 4, 5, 2]$ as a row vector. Extend the continued fraction representation of $5/2$ to a row vector of the same length by adding zeros at the front, $[0, 0, 0, 0, 2, 2]$. Take the difference between both vectors, $[3, 2, 6, 4, 3, 0]$. Divide the even-indexed (note that the leading terms is indexed by 0) terms by 3 and the odd-indexed terms by 2, $[1, 1, 2, 2, 1, 0]$. Corollary 3 states that the sum of the terms in this vector gives the depth of $2147/620$. The terms

also show that $2147/620 = R_v L_u R_v^2 L_u^2 R_v(5/2)$. In particular, since

$$(14) \quad R_v L_u R_v^2 L_u^2 R_v = \begin{bmatrix} 187 & 606 \\ 54 & 175 \end{bmatrix},$$

then $a = 187$, $b = 606$, $c = 54$, and $d = 175$ in Lemma 3 for this case.

When $u = v = 1$, the above discussion shows that every positive rational number appears in the original Calkin-Wilf tree (see [4]).

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